International Journal of Theoretical Physics, Vol. 38, No. 4, 1999

Black Hole Formation by Canonical Dynamics of Gravitating Shells: An Equatorial View

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Received September 15, 1998

Gravitating shells lead to simple minisuperspace models of black hole formation by gravitational collapse of matter. I interpret here the Hájíček–Kijowski variational principle for spacetime with a shell as a Dirac–ADM action principle along a timelike foliation including the shell as a leaf. By reducing this action by spherical symmetry, I obtain the Hamiltonian constraint of a collapsing dust shell and use it as a prelude to canonical quantization.

1. SOUTHERN AND NORTHERN CONE VIEWPOINTS

Mnemonics are wonderful devices for remembering elusive trivialities. As I was leaving Salt Lake City for Bariloche, the crescent moon hung like a letter C on the evening sky. Almost every language has a mnemonic for remembering the phases of the Moon by their resemblance to some letters of the alphabet. In Czech, the Moon which makes C on the sky Couvá, it is waning. Latin, however, has the mnemonic upside down:

Perfida luna: **D**ecrescendo Crescit, **C**rescendo Decrescit.

Deceitful Moon: When she pretends to wane, she waxes, when she pretends to wax, she wanes. The boldface letter with which the Latin word describing the phase begins is exactly opposite to the sans serif shape of the Moon on the sky. The Moon is double-crossing us.

Last evening, as I was returning from the dinner to my room, I noticed something strange: the waning moon which day before had hung like C on the sky in Salt Lake City was shaped like D on the Bariloche sky. Mnemonics should depend on the hemisphere. Did the Romans when they turned Italians and emigrated to Argentina conclude that the Moon became trustworthy? I

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doubt it. They probably concluded that the Moon is as deceitful as ever and that she is merely double-double crossing them.

I want to thank the organizers of this meeting on behalf of all of us from the North for the chance to view celestial physics from the Southern perspective. In spacetime, such a switch would correspond to turning the lightcone upside down. I my talk, I want to explore a less dramatic change of direction in the sublunary physics of canonical gravity. What I have to say is associated with viewing the production of a black hole spacetime by spherically collapsing shell not as a vertical evolution from the past to the future, but rather as a horizontal canonical process in the spacelike direction perpendicular to the shell history. I, a Northerner lecturing about dynamics in the Southern Cone, am offering to meet my listeners half the way: Let us view together the spacetime dynamics as if it occurs along the equator.

2. MATTER SHELLS AS MODELS OF THE GRAVITATIONAL COLLAPSE

Evaporation of black holes by Hawking radiation is a semiclassical effect which should ultimately be described by quantum gravity. A complementary process is the formation of a quantum black hole by the gravitational collapse of quantized matter. While modeling the Hawking radiation in canonical quantum gravity requires quantum field theory (a midisuperspace model), gravitational collapse can be studied on quantum systems with a finite number of degrees of freedom (minisuperspace models). The simplest example of such models are spherically symmetric thin shells.

One can guess a Lagrangian which yields the known equations of motion of the shell and use it for quantizing that motion. For a dust shell, this was done by Hájíček, Kay and Kuchař [1] and for different types of shells by a number of people in a number of ways.[2] An uneasiness with such a shortcut to quantization arises from ambiguities inherent in the inverse problem of the calculus of variations [4].[3] One can limit such ambiguities by additional requirements [7], but the uneasiness persists. To root the super-Hamiltonian of a highly specialized system in the bedrock of first principles, one should be able to obtain it by consistently reducing a variational principle of the general system.

[2] Spherical vacuum shells are treated, e.g., in refs. 2. The quantization of false vacuum bubbles is discussed using the Dirac formalism and the WKB approximation in ref. 3. The treatment of Kraus and Wilczek [20] follows the methods of ref. 3.

[3] Refs. 1 and 5 contain four examples of very different Lagrangians that all describe the same system of a spherically symmetric massive dust shell. They are classically equivalent in the sense that there are smooth transformations between them, but the quantum theories are not unitarily equivalent. An example of two Lagrangians that lead to the same equations of motion but are even classically inequivalent is given in ref. 6.

Hájíček and Kijowski [8] have recently formulated such a variational principle which yields in one stroke the dynamics of the matter within a shell, the dynamics of the shell in a surrounding spacetime, and geometrodynamics of that spacetime. For a spherically symmetric shell, neither the flat interior nor the curved exterior of the shell have their own dynamical degrees of freedom. The Schwarzwschild mass of the exterior is determined from the dynamics of the matter variables (reduced to a finite number by spherical symmetry) and the dynamics of the intrinsic geometry of the shell—the change of its area coordinate R. Spherical reduction of the Hájíček–Kijowski action then yields a *unique* shell action which describes the internal dynamics of the shell's matter and geometry. This can be taken as a basis for the Dirac constraint quantization of the gravitational collapse.

I shall interpret here the Hájíček–Kijowski variational principle as a Dirac–ADM action principle along a *timelike* foliation including the shell as a leaf. I show how this formulation simplifies the spherical reduction to the shell action. I also show how to write the Lagrangian of the simplest conceivable material of the shell, an incoherent dust, in a form in which the rest mass density of the dust and proper time along the dust worldlines appear as conjugate canonical variables. The quantum constraint for a spherically symmetric dust shell then naturally yields a Schrödinger equation for the dynamics of the shell geometry.

3. SHELL DYNAMICS FROM ISRAEL'S JUNCTION CONDITION

Any contemporary treatment of the shell motion in general relativity is likely to start from a junction condition found by Dautcourt [9] and cast in an elegant geometric form by Israel [10].

Envisage a shell of matter which moves in an empty Ricci-flat spacetime. Figure 1 shows a closed shell S which divides the spacetime $(\mathcal{M}, \mathsf{g})$ into an interior $(\mathcal{M}^-, \mathsf{g}^-)$ and exterior $(\mathcal{M}^+, \mathsf{g}^+)$ regions. Israel's description of how to join $\mathcal{M}^-$ to $\mathcal{M}^+$ across the shell is entirely given in terms of geometric quantities. Coordinate patches $X^{\pm\mathsf{a}}$, $\mathsf{a} = 0, 1, 2, 3$, and x^a, $a = 0, 1, 2$, which are used in the spacetimes $\mathcal{M}^\pm$ and on the shell S may be arbitrary and mutually independent. The way in which the shell moves in the interior and exterior spacetimes is described by the embedding functions

$$X^{\pm\mathsf{a}} = X^{\pm\mathsf{a}}(x^a) \tag{1}$$

Israel's first condition (the continuity condition) is the requirement that the intrinsic metric $g_{ab}(x)$ on S be the same whether it is induced by the interior spacetime metric $\mathsf{g}^-_{\mathsf{ab}}$ or the exterior spacetime metric $\mathsf{g}^+_{\mathsf{ab}}$:

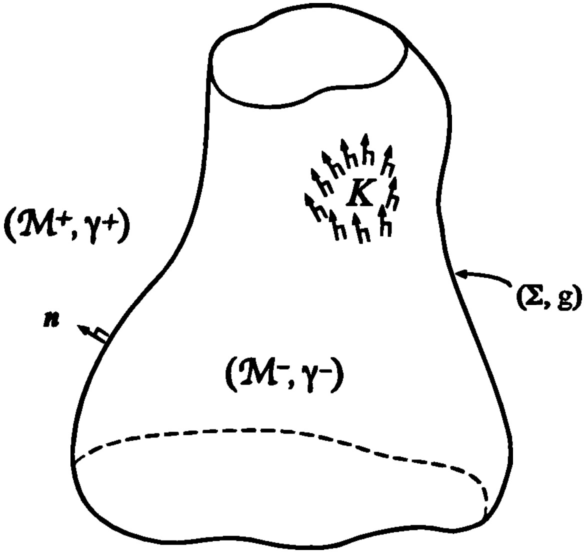

Fig. 1. A closed shell (Σ, g) with interior $(\mathcal{M}^-, \gamma^-)$ and exterior $(\mathcal{M}^+, \gamma^+)$. The normal $\mathbf{n}$ points from $\mathcal{M}^-$ to $\mathcal{M}^+$. The extrinsic curvature K_{ab} characterizes the bending of the shell in $\mathcal{M}$.

$$\gamma^-{}_{\alpha\beta}(X^-(x))X^{-\alpha}{}_{,a}(x)X^{-\beta}{}_{,b}(x) = g_{ab}(x) = \gamma^+{}_{\alpha\beta}(X^+(x))X^{+\alpha}{}_{,a}(x)X^{+\beta}{}_{,b}(x) \tag{2}$$

The matter on the shell is characterized by an energy-momentum tensor $T^{ab}(x)$ whose covariant divergence with respect to the intrinsic metric vanishes:

$$T^{ab}{}_{|b}(x) = 0 \tag{3}$$

The second junction condition tells us how the extrinsic curvature

$$K_{ab}(x) := -n_{\alpha;\beta} X^{\alpha}_{,a} X^{\beta}_{,b} \tag{4}$$

changes when we pass from the interior of the shell to its exterior. It connects the jump [ ] of the contravariant tensor density

$$p^{ab}(x) := |g|^{1/2}(Kg^{ab} - K^{ab}) \tag{5}$$

constructed from the intrinsic metric g_{ab} and extrinsic curvature K_{ab} of Σ to the energy-momentum tensor on the shell,[4]

[4] Before going on, I need to explain my use of units. In canonical geometrodynamics, to avoid inconvenient factors in the Hamiltonian constraint, one introduces natural units in which $c = 1$ and the Einstein constant of gravitation κ is put equal to 1/2. I use these units in all general equations of this paper. Thus, the factor 1/2 on the right side of Eq. (6) is just the Einstein constant κ. However, once I reduce the canonical action by spherical symmetry, such a choice of natural units brings inconvenient 4π factors by integrations over the unit sphere. To avoid such factors, I then switch to natural units in which the Newton constant of gravitation G is put equal to 1: $G = 1$.

$$-[p^{ab}] := -(p_-{}^{ab} - p_+{}^{ab}) = \frac{1}{2} |g|^{1/2} T^{ab} \tag{6}$$

On a spacelike hypersurface, the expression (5) represents the Dirac–ADM (Arnowitt, Deser, and Misner) gravitational momentum. Of course, the shell history S is a timelike rather than a spacelike hypersurface. Nevertheless, the canonical perspective is a handy tool in all our considerations.

The Israel junction condition implies both how the matter moves within the shell and how the shell moves in the embedding spacetime. The motion of the matter is given by the conservation law (3), which follows from the junction condition (6) by virtue of the Gauss–Coddazi equation

$$p^{ab}{}_{|b} = 0 \tag{7}$$

That the motion of the shell also follows from the Israel junction condition is an example of the general fact that Einstein's field equations imply the equations of motion of the field 'singularities' [11]. We shall see later how this happens in the context of the Hamiltonian theory of a spherical shell.

In spite of the fact that the Israel junction condition (6) tells us all that we need to know about the motion of the shell and its matter, it does not tell us enough to quantize this motion. Quantization does not work at the level of equations of motion, but requires the knowledge of the Lagrangian or the Hamiltonian of the system. The problem with the Israel formulation as a basis for quantum theory is that *neither* the dynamics of matter within the shell *nor* the junction condition itself are obtained from an action principle.

4. HILBERT ACTION AND THE DIRAC–ADM ACTION

To explain how to solve these difficulties, I need to remind you of two alternative forms of the action principle for Einstein's equations. It has been known *ab urbe condita* [12] that the vacuum Einstein equations follow from the Hilbert action

$$S^{H}[\mathsf{g}] = \int d^4X\, |\mathsf{g}|^{1/2} \mathbf{R}(x; \mathsf{g}] \tag{8}$$

by varying the spacetime metric g_{ab} in a compact region of spacetime. It took, however, some time to understand what happens if the variation does not vanish at the boundary $\mathsf{S} = \partial\mathcal{M}$ of region $\mathcal{M}$ [13]. In this case, the variation of the Hilbert action acquires a boundary term which is the Liouville form in the variation of the gravitational momentum:

$$\mathsf{d}S^{H}[\mathsf{g}] = -\int_{\mathcal{M}} d^4X\, |\mathsf{g}|^{1/2} G^{\mathsf{ab}} \mathsf{d}\mathsf{g}_{\mathsf{ab}} - \int_{\mathsf{S}} d^3x\, g_{ab} \mathsf{d}p^{ab} \tag{9}$$

The spacetime metric in the Hilbert action must be varied so that p^{ab} on the boundary (i.e., the *normal derivative* of the induced metric g_{ab}) is kept fixed. Otherwise, the variation of the momentum p^{ab} on S would force on us a blatantly unnatural 'natural' boundary condition $g_{ab} = 0$. On the other hand, the variation does not need to limit in any way the change of the induced metric g_{ab}. The Hilbert action is thus geared to the Neumann boundary conditions on S.

To get the form of the action adapted to the Dirichlet boundary conditions, one needs to amend S^{H} by a surface term:

$$S^{\mathrm{G}}[\mathsf{g}] := S^{\mathrm{H}}[\mathsf{g}] + \int_{\mathsf{S}} d^3x\, p \tag{10}$$

The terms obtained by varying the trace $p = g_{ab}p^{ab}$ of the gravitational momentum exactly cancel the boundary term in the variation (9) of S^{H} and introduce the Dirichlet type boundary term in the variation of the gravitational action $S^{\mathrm{G}}[\mathsf{g}]$:

$$\mathsf{d}S^{\mathrm{G}}[\mathsf{g}] = -\int_{\mathcal{M}} d^4X\, |\mathsf{g}|^{1/2} G^{\mathsf{ab}} \mathsf{d}\mathsf{g}_{\mathsf{ab}} + \int_{\mathsf{S}} d^3x\, p^{ab} \mathsf{d}g_{ab} \tag{11}$$

To avoid an unwanted 'natural' boundary condition $p^{ab} = 0$, one must now vary the spacetime metric close to the boundary so that the induced metric g_{ab} on S is kept fixed. In exchange, no *a priori* limitation needs to be imposed on how the normal derivatives p^{ab} of the induced metric can change. This is why the new form (10) of the action is suitable for deriving the Israel junction condition (6) for the momenta p^{ab} from a variational principle.

The gravitational action (10) is well known to people working in canonical gravity. When expressed along a foliation of $\mathcal{M}$ by spacelike hypersurfaces S_t, it is exactly the Lagrangian form of the Dirac–ADM action.

5. ISRAEL'S JUNCTION CONDITION FROM AN ACTION PRINCIPLE

I now want to apply a variational principle to a boundary which is not an arbitrarily prescribed hypersurface, but which is physically delineated by the motion of a thin layer of matter. Such matter may be phenomenological (dust, fluid, or elastic medium) or it may be a more fundamental tensor field (scalar field, electromagnetic field, or combination of coupled fields) ordinarily considered in a four-dimensional spacetime.

The action describing the propagation of phenomenological matter or fields within the shell is supposed to be given in a Lagrangian form

$$S_{\mathsf{S}}^{\mathrm{M}} = \int_{\mathsf{S}} d^3x \, L^{\mathrm{M}}[g, \mathsf{F}] \tag{12}$$

The Lagrangian $L^{\mathrm{M}}[g, \mathsf{F}]$ is a scalar density under Diff$\mathsf{S}$ constructed from the intrinsic metric g_{ab} of S, the matter fields F, and the derivatives of g_{ab} and F on S up to a finite order. I explicitly assume that the propagation of the fields within S depends only on the intrinsic metric of S, not on the bending of S in $\mathcal{M}^-$ or $\mathcal{M}^+$.

By varying $S_{\mathsf{S}}^{\mathrm{M}}$ with respect to the fields, we obtain the field equations. As in a four-dimensional spacetime, the symmetric energy-momentum tensor of matter fields is given by the variational derivative of the matter action with respect to the intrinsic metric on S:

$$T^{ab}(x) := 2|g|^{-1/2} \frac{\mathsf{d} S_{\mathsf{S}}^{\mathrm{M}}}{\mathsf{d} g_{ab}(x)} \tag{13}$$

Dust coupled to gravity is an especially handy conceptual tool for interpreting quantum dynamics [14]. After spherical minisuperspace reduction, the Hamiltonian constraint for the dust shell naturally assumes the form of a Schrödinger equation. For the Lagrangian of the shell, I take the three-dimensional version of a Lagrangian introduced by Brown [15]. The dust shell is described by six scalar fields which I call

$$\mathsf{T}(x),\ Z^k(x);\quad \mathsf{r}(x),\ W_k(x), \qquad \text{with} \quad k = 1, 2 \tag{14}$$

The functions $\mathsf{T}(x)$ and $Z^k(x)$ are assumed to be independent, i.e., their gradients $\mathsf{T}_{,a}(x)$, $Z^k_{,a}(x)$ must form a cobasis in $T^*\mathsf{S}$. The Lagrangian has the form

$$L^{\mathrm{D}} = -\frac{1}{2} |g|^{1/2} \mathsf{r} \left(g^{ab} U_a U_b + 1\right) \tag{15}$$

where the velocity covector U_a is a Pfaff form constructed from the Clebsch potentials $\mathsf{T}(x)$, $Z^k(x)$, and $W_k(x)$:

$$U_a := -\mathsf{T}_{,a} + W_k Z^k{}_{,a} \tag{16}$$

The equations of motion obtained by varying the potentials (14) shed light on their physical meaning: The scalars $Z^k(x)$ are two comoving coordinates labeling the dust worldlines within the shell, $\mathsf{T}(x)$ is the proper time along those worldlines, $W_k(x)$ are the normal projections of the velocity U^a onto surfaces of constant T expressed in the comoving basis, and $\mathsf{r}(x)$ is the surface density of the rest mass of the dust on the shell. The Euler–Lagrange equations imply that flowlines of U^a are geodesics of the shell metric g_{ab} and that the rest mass satisfies the continuity equation $(\mathsf{r}\, U^a)_{|a} = 0$. Moreover, the variation of the shell action gives by Eq. (13) the standard energy-momentum tensor

$$T^{ab} = \rho U^a U^b \tag{17}$$

of the dust.

This solves the first problem presented by the Israel junction condition as a would-be basis for quantization: The action (12) in general and the dust shell action (15) in particular provide all information about the propagation of matter fields on Σ. However, we still lack a variational principle determining the motion of the shell in the surrounding spacetime, i.e., giving the rules according to which (Σ, g) is embedded in $(\mathcal{M}^-, \gamma^-)$ and $(\mathcal{M}^+, \gamma^+)$. Such a principle was introduced by Hájíček and Kijowski [8]. I formulate it here directly in terms of the Dirac–ADM gravitational action:

The fields $\Phi(x)$ on the shell, the shell metric g_{ab} on Σ, and the metrics $\gamma^{\pm}{}_{\alpha\beta}(X^{\pm})$ in $\mathcal{M}^{\pm}$ extremize the action

$$S[\gamma^{\pm}, g, \Phi] = S^G_+[\gamma^+] + S^G_-[\gamma^-] + S^M_\Sigma[g, \Phi] \tag{18}$$

when g and $\gamma^{\pm}$ are varied under the auxiliary *continuity condition* (2).

The variational formula (11) for the gravitational action enables me to explain immediately how this principle works. The variation of Eq. (18) yields

$$\begin{aligned} \delta S = &- \int_{\mathcal{M}^-} d^4X^- \, |\gamma^-|^{1/2} G^{-\alpha\beta}(X^-) \delta\gamma^-{}_{\alpha\beta}(X^-) \\ &- \int_{\mathcal{M}^+} d^4X^+ \, |\gamma^+|^{1/2} G^{+\alpha\beta}(X^+) \delta\gamma^+{}_{\alpha\beta}(X^+) \\ &+ \int_\Sigma d^3x \, [p^{ab}(x)] \delta g_{ab}(x) + \int_\Sigma d^3x \, \frac{1}{2} |g|^{1/2} T^{ab}(x) \delta g_{ab}(x) \end{aligned} \tag{19}$$

(Because we agreed to orient both normals $\mathbf{n}^{\pm}$ from $\mathcal{M}^-$ to $\mathcal{M}^+$, the normal $\mathbf{n}^+$ is oriented *into* $\mathcal{M}^+$ and p^{+ab} thus appears with the minus sign in the jump $[p^{ab}(x)] = p^{ab}{}_-(x) - p^{ab}{}_+(x)$.)

The variation of the spacetime metric in $\mathcal{M}^{\pm}$ yields the Einstein law in vacuum regions:

$$G^{\pm\alpha\beta}(X^{\pm}) = 0 \tag{20}$$

The variation of the intrinsic metric g_{ab} on Σ dutifully reproduces the Israel junction condition (6).

So far, the action principle (18) has the Lagrangian form and geometry is specified by the metrics $\gamma^{\pm}$ and g. The momentum variables are mere abbreviations for the expressions (4)–(5). However, I have already noticed that the gravitational action S^{G} is naturally connected with the Dirac–ADM action. This suggests that one can incorporate the shell history $X(x)$ into a foliation $X_r(x)$ of spacetime by *timelike* hypersurfaces

$$X_r\colon\quad \mathbb{R} \times \mathsf{S} \to \mathcal{M} \qquad \text{by} \qquad r \in \mathbb{R}, \quad x \in \mathsf{S} \mapsto X = X(r, x) \in \mathcal{M} \tag{21}$$

and vary the action in the canonical form. I freely use this point of view in the following discussion. To cross its t's and dot its i's requires, of course, a more detailed presentation [16].

6. SPHERICAL REDUCTION

I now reduce the shell action (18) by spherical symmetry. My approach is distinguished from previous treatments of the same problem by reducing the Dirac–ADM action along a *timelike* foliation including the shell. In this way, one gains a fresh insight into why and how the dynamics is carried not by the whole spacetime, but merely by the shell.

The three kinds of objects which need to be reduced by spherical symmetry are *the shell geometry, the spacetime geometry*, and *matter fields* on the shell. Their reduced form allows me to proceed with the spherical reduction of the *matter shell action* and *canonical gravitational action*.

6.1. Spherical Reduction of the Shell Geometry

I label the events on a history S of a spherically symmetric shell by arbitrary coordinates j^m, $m = 1, 2$, of the homogeneous S^2 sections [say, by the standard spherical coordinates $\mathsf{j}^m = (\mathsf{u}, \mathsf{w})$], and by an arbitrary monotonically growing time label t along the j^m = const generators. The geometry of S then takes the form

$$d\mathsf{s}^2 = -\mathsf{L}^2(t)\, dt^2 + R^2(t)\, d\mathsf{V}^2 \tag{22}$$

where $d\mathsf{V}^2 = \mathsf{V}_{mn} d\mathsf{j}^m d\mathsf{j}^n$ is the geometry of unit sphere. The intrinsic metric $g_{ab}(x)$ of the shell is thereby reduced to two functions, $R(t)$ and $\mathsf{L}(t)$, of a single coordinate, the label time t. The variable R is determined by the instantaneous proper area of the shell. When one changes t in the line element (22), $R(t)$ transforms as a scalar. The second metric coefficient, $\mathsf{L}(t)$, determines the proper time

$$d\mathsf{t} = \mathsf{L}(t)\, dt \tag{23}$$

along generators of the shell history. The proper time t, like the label time t, grows from the past to the future and hence $\mathsf{L}(t) > 0$. When one transforms t, $\mathsf{L}(t)$ behaves as a scalar density.

6.2. Spherical Reduction of the Matter Shell Fields

Superficially, it may seem that spherical symmetry demands that the six dust potentials (14) do not depend on the angular variables j^m, and hence reduce to some arbitrary functions of the time label t. However, that would violate the condition that the reduced covectors $\mathsf{T}_{,a}$, $Z^k{}_{,a}$ be linearly independent. For dust to be in a spherically symmetric fall, its particles must be pegged to fixed locations j of the spherical shell. To achieve that, one should take $Z^k(x)$ to be fixed functions of coordinates j^m and prohibit their dependence on t: $Z^k = Z^k(\mathsf{j}^m)$, e.g., $Z^k = \mathsf{j}^k$. These functions cannot be varied and hence lose their role of field variables in the reduced action. Because dust particles cannot move along the sphere, but only radially fall with it, the projection of their velocity into the sphere must vanish: $W_k = 0$. I conclude that the appropriate reduction of the dust variables is

$$\mathsf{T} = \mathsf{T}(t), \quad \mathsf{r} = \mathsf{r}(t); \qquad Z^k = Z^k(\mathsf{j}^m), \quad W_k = 0 \tag{24}$$

The dust Lagrangian (15) then depends only on two scalar variables $\mathsf{T}(t)$ and $\mathsf{r}(t)$.

6.3. Spherical Reduction of the Dust Shell Action

The reduced form (24) of the dust variables leaves $U_t = -\dot{\mathsf{T}}(t)$ as the only surviving component of the Pfaff form (16). Because

$$|g|^{1/2} = |\mathsf{V}|^{1/2} R^2 \mathsf{L} \qquad \text{and} \qquad g^{tt} = -\mathsf{L}^{2} \tag{25}$$

the dust Lagrangian reduces to

$$L^{\mathrm{D}}(\mathsf{L}, R; \mathsf{r}, \dot{\mathsf{T}}) = \frac{1}{2}\, \mathsf{r}\, R^2 |\mathsf{V}|^{1/2} (\mathsf{L}^{-1}\dot{\mathsf{T}}^2 - \mathsf{L}) \tag{26}$$

When I integrate it over the unit sphere and replace the multiplier $\mathsf{r}(t)$ by the total rest mass multiplier

$$\mathsf{M}(t) := 4\mathsf{p} R^2(t) \mathsf{r}(t) \tag{27}$$

I arrive at the dust shell action

$$S^{\mathrm{D}}_{\mathrm{S}}[\mathsf{L}; \mathsf{M}, \mathsf{T}] = \int dt\, \frac{1}{2} \mathsf{M} (\mathsf{L}^{-1}\dot{\mathsf{T}}^2 - \mathsf{L}) \tag{28}$$

6.4. Spherical Reduction of the Spacetime Gravitational Action

Spherical reduction of the shell geometry replaced the intrinsic metric g_{ab} by two functions L and R of a single variable, the label time t. Similarly, spherical reduction of the spacetime geometry replaces g_{ab} by four functions of two variables, the time label t and a radial label r. When we substitute these functions into the gravitational action and vary them so that they match a given shell metric L, R, we get the vacuum Einstein equations in $\mathcal{M}^{\pm}$. Their solution gives a Minkowski spacetime inside a collapsing shell, and a Schwarzschild solution corresponding to some constant Schwarzschild mass M outside the shell. This procedure for obtaining the Schwarzschild solution from the action principle reduced by spherical symmetry essentially follows the outlines devised by Weyl [17].

Substitute now this solution of the bulk Einstein equations into the gravitational action and vary the intrinsic metric g_{ab} on S. Because the bulk equations are satisfied, the variational formula (19) retains only the boundary term

$$\mathsf{d}S^{G} = \int_{\mathsf{S}} d^3x\, [p^{ab}]\mathsf{d}g_{ab} \tag{29}$$

$$= \int_{\mathsf{S}} dt\, ([\mathsf{P}_{\Lambda}]\mathsf{dL} + [\mathsf{P}_{R}]\mathsf{d}R) \tag{30}$$

Because the metric is limited by spherical symmetry, the general formula (29) is reduced to variations of the two metric coefficients L and R multiplied by the corresponding momenta $[\mathsf{P}_{\Lambda}]$ and $[\mathsf{P}_{R}]$.

I can use the bulk solutions (the Minkowski and Schwarzschild spacetimes) to obtain the coefficients P_{Λ} and P_{R}. To explain their meaning, I evoke the canonical version of the gravitational action along a timelike foliation including the shell history:

The momentum constraint (7) is reduced by spherical symmetry to the statement

$$\mathsf{P}_{R}\dot{R} - \mathsf{L}\dot{\mathsf{P}}_{\Lambda} = 0 \tag{31}$$

To obtain Eq. (31) does not require any calculation: the form of its left side is dictated by the fact that R is a scalar and L a scalar density under diffeomorphisms of t which the constraint (31) is supposed to generate [18]. From Eq. (31) it follows that the first momentum coefficient completely determines the second:

$$\mathsf{P}_{R} = V^{-1}\dot{\mathsf{P}}_{\Lambda} \tag{32}$$

The scalar

$$V := \dot{R}/\mathsf{L} \tag{33}$$

is dynamically so important that it deserves a name. It is the rate of change of the invariantly defined area coordinate R with proper time along the shell history S. I call V the *proper velocity* of the shell.

The remaining coefficient Π_Λ is a key to the shell dynamics. Its meaning can be inferred from Eq. (5) restricted by spherical symmetry: One learns that $8\pi \Pi_\Lambda$ is the rate at which the area of the 2-spheres grows in the direction **n** normal to the shell history Σ, i.e.,

$$\Pi_\Lambda = \frac{1}{2} \partial_{\mathbf{n}} R^2 \tag{34}$$

In the Schwarzschild solution, this rate can be expressed in terms of the area coordinate R and the proper velocity V of the shell. I write the resulting formula in the static quadrant of the Kruskal diagram where R increases in the direction of **n**:

$$\Pi_\Lambda = R\sqrt{F + V^2} \tag{35}$$

The function

$$F(R, M) := 1 - 2M/R \tag{36}$$

is an abbreviation for the combination of R and M variables which naturally appears in the Schwarzschild solution written in curvature coordinates. (In the dynamical quadrants $F < 0$, R can either decrease or increase along **n**. To keep track of the signs and cover all relevant quadrants is vital, but I will write here the explicit formulas only for the static quadrant.) In the Minkowski spacetime interior of the shell $F = 1$.

7. REDUCTION OF SPACETIME ACTION TO SHELL ACTION

I am now going to argue that the boundary term (30) can be expressed as the variation of a gravitational shell action

$$S^G_\Sigma[\Lambda, R] = \int_\Sigma dt\, L^G_\Sigma(\Lambda, R, \dot{R}) = \int_\Sigma dt\, \Lambda L(R, V) \tag{37}$$

which is a functional of the intrinsic metric Λ, R of the shell. In other words, I claim that the variational differential form (30) is *exact*.

The shell action S^G must be invariant under diffeomorphisms of the label time t, i.e., the Lagrangian $L^G_\Sigma(\Lambda, R, \dot{R})$ must be a homogeneous function of the variables Λ and $\dot{R}$. The most general homogeneous Lagrangian is $\Lambda L(R, V)$, which explains the last form of Eq. (37).

By comparing the expressions (30) and (37), I get two equations for L^G_Σ:

$$E_\Lambda := L^G_{\Sigma,\Lambda} = L - L_{,V} V = R[\sqrt{F + V^2}] \tag{38}$$

and

$$E_R := L^G_{S,R} - (L^G_{S,\dot{R}})^{\bullet} = V^{-1}[\mathsf{P}_\Lambda]^{\bullet} \tag{39}$$

where E_Λ and E_R are the Euler–Lagrange expressions of L^G_S. Because the Lagrangian L^G_S is homogeneous, the Euler identity implies that the second equation follows from the first. Hence, the problem reduces to the solution of a single differential equation (38), which amounts to a familiar task of the Legendre transform procedure: To find the Lagrangian L from its energy function $\mathsf{L}_{,V}V - \mathsf{L} = -R[\sqrt{F + V^2}]$. This task has the solution

$$\mathsf{L} = R[\sqrt{F + V^2} - V \sinh^{-1}(F^{-1/2}V)] \tag{40}$$

which again holds in the static quadrant of the Kruskal diagram. (In dynamical quadrants, one needs to replace sinh by cosh and change some signs.)

I have thus showed that the boundary term (30) is the variation of the gravitational shell action (37) with the scalar Lagrangian (40).

8. TOTAL SHELL ACTION

The total shell action $S_\Sigma[\mathsf{L}, R; \mathsf{T}, \mathsf{M}]$ consists of the gravitational part (37), (40) and the dust part (28):

$$S_\Sigma[\mathsf{L}, R; \mathsf{T}, \mathsf{M}] = S^G_S[\mathsf{L}, R] + S^D_S[\mathsf{L}; \mathsf{T}, \mathsf{M}] \tag{41}$$

Its variation in the metric variables L, R and the matter variables T, M gives the internal dynamics of geometry and matter of the shell. In particular, by varying T, I check that M is a constant of motion, by varying M, I check that the lapse function is the rate of change of the proper time with the label time, $\mathsf{L} = \dot{\mathsf{T}}$, and by varying L, I get the conservation law

$$\sqrt{1 + V^2} - \sqrt{F + V^2} = \frac{\mathsf{M}}{R} \tag{42}$$

By integrating Eq. (42), I could find how the area coordinate changes with proper time: $R = R(\mathsf{T})$.

Unlike the rest mass $\mathsf{M}(t)$ of the shell, the Schwarzschild mass enters the shell action S_Σ as a constant. Quite remarkably, one can turn M into a dynamical variable $M(t)$ when extending the action S_Σ by yet another variable $T(t)$:

$$S_\Sigma[\mathsf{L}, R; \mathsf{T}, \mathsf{M}; T, M] = -\int dt\, M\dot{T} + S^G_S[\mathsf{L}, R, M] + S^D_S[\mathsf{L}; \mathsf{T}, \mathsf{M}] \tag{43}$$

By varying the extended action with respect to T, I show that M is actually a constant of motion. By varying it with respect to M, I get the relation

$$\dot{T} = \mathsf{L} F^{-1} \sqrt{F + V^2} \tag{44}$$

which reveals the meaning of the variable $T(t)$: Eq. (44) is a valid equation for the rate of change of the Killing time T of the exterior Schwarzschild solution along the shell history. The extended action thus gives not only the internal dynamics of the shell, but also its equation of motion in the exterior spacetime. This illustrates my previous statement that the Israel junction condition leads to the equations of motion of the shell.

To quantize the internal and external dynamics of the shell, I must cast the extended shell action in canonical form. Let me do it first for the dust part. By introducing the momentum P conjugate to T and performing the Legendre dual transformation, I get

$$S^{D}_{\mathsf{S}}[\mathsf{L}, \mathsf{M}; \mathsf{T}, \mathsf{P}] = \int dt \left(\mathsf{P}\dot{\mathsf{T}} - \mathsf{L}\, \frac{1}{2} (\mathsf{M}^{-1}\mathsf{P}^2 + \mathsf{M}) \right) \tag{45}$$

The variables L and M are Lagrange multipliers. By varying the action with respect to M, I find that the rest mass M is identical with the momentum P: $\mathsf{M} = \mathsf{P}$. By identifying P with M in Eq. (45), I cast the dust action into canonical form

$$S^{D}_{\mathsf{S}}[\mathsf{L}; \mathsf{T}, \mathsf{M}] = \int dt\, (\mathsf{M}\dot{\mathsf{T}} - \mathsf{L}\mathsf{H}^{D}) \tag{46}$$

in the conjugate variables T and M. The Hamiltonian of this action is proportional to M:

$$\mathsf{H}^{D}(\mathsf{T}, \mathsf{M}) = \mathsf{M} \tag{47}$$

Similarly, by introducing the momentum P canonically conjugate to R and performing the Legendre dual transformation, I could derive the canonical form of the gravitational action:

$$S^{G}_{\mathsf{S}}[\mathsf{L}; R, P; M, T] = \int dt\, (P\dot{R} - M\dot{T} - \mathsf{L}\mathsf{H}^{G}(R, P)) \tag{48}$$

The gravitational super-Hamiltonian in the static quadrant of the Kruskal diagram is

$$\mathsf{H}^{G}(R, P) = -R \left[\quad \sqrt{1 + F - 2F^{1/2} \cosh R^{-1}P} \right] \tag{49}$$

It has been previously obtained by a different reduction process by Hájíček [19].

The total shell action thus has the generalized Hamiltonian form

$$S_\Sigma[\mathsf{L}\,;\, R,\, P;\, \mathsf{T},\, \mathsf{M};\, T,\, M] = S^{\mathrm{G}}_{\mathrm{S}}[\mathsf{L}\,;\, R,\, P;\, T,\, M] + S^{\mathrm{D}}_{\mathrm{S}}[\mathsf{L}\,;\, \mathsf{T},\, \mathsf{M}]$$

$$= \int dt\, (P\dot{R} + \mathsf{M}\dot{\mathsf{T}} - M\dot{T} - \mathsf{L}\mathsf{H}) \tag{50}$$

The variation of the lapse multiplier L constrains the super-Hamiltonian

$$\mathsf{H} = \mathsf{H}^{\mathrm{D}}(\mathsf{T},\, \mathsf{M}) + \mathsf{H}^{\mathrm{G}}(R,\, P;\, M) = \mathsf{M} - R\left[\sqrt{1 + F - 2F^{1/2}\cosh R^{-1}P}\right] \tag{51}$$

to vanish.

9. INNER AND OUTER QUANTUM GEOMETRODYNAMICS OF A DUST SHELL

The shell action (50)–(51) contains two times, the proper time T and the outer Killing time T along the shell; let me call it a *double time action*. Both of these times are ignorable (only their derivatives $\dot{\mathsf{T}}$ and $\dot{T}$ appear in the canonical Lagrangian), and hence the conjugate momenta M and $-M$ (the rest mass of the shell and the Shwarzschild mass of the outer spacetime) are constants of motion. The Dirac constraint quantization of the motion of the shell calls for the replacement of the conjugate canonical variables by operators, their substitution into the super-Hamiltonian, and the imposition of the Hamiltonian constraint as an operator restriction on the state function C:

$$\dot{\mathsf{H}}\mathsf{C} = 0 \tag{52}$$

Because the super-Hamiltonian (51) is linear in the proper mass M, the quantum Hamiltonian constraint (52) written in the $(\mathsf{T},\, R,\, M)$ representation takes the form of a Schrödinger equation in the proper time T:

$$i\,\frac{\partial \mathsf{C}(\mathsf{T};\, R,\, M)}{\partial \mathsf{T}} = \mathsf{H}^{\mathrm{G}}(\hat{R},\, \hat{P},\, \hat{M})\mathsf{C}(\mathsf{T};\, R,\, M) \tag{53}$$

If I were able to factor order the operators $\hat{R}$, $\hat{P}$, and $\hat{M}$ so that the gravitational super-Hamiltonian $\mathsf{H}^{G}(\hat{R},\, \hat{P},\, \hat{M})$ would be a self-adjoint operator, the Schrödinger equation would give a clear probabilistic interpretation to the variables R, P, T, and M. Unfortunately, this is not an entirely straightforward task because of the positivity restrictions on the domains of the variables R and M and the rather complicated form of H, which requires that one define the cosh and square root operations by spectral analysis. However, modulo these technical difficulties, our shell model can assign a probabilistic interpretation to the *intrinsic geometry operator* of the shell history

$$d\hat{\mathsf{s}}^2 = -d\mathsf{T}^2 + \hat{R}^2(\mathsf{T})\, d\mathsf{V}^2 \tag{54}$$

in the Heinsenberg picture.

In particular, one can ask how the probability distributions of the Heisenberg operators $\hat{M}(\mathsf{T})$, $\hat{R}(\mathsf{T})$, or $\hat{F} := 1 - 2\hat{M}(\mathsf{T})/\hat{R}(\mathsf{T})$ change with proper time T. Predictably, the probability distribution of the $\hat{M}(\mathsf{T})$ operator should not change at all because the Schwarzschild mass operator is a quantum constant of motion. By following how the remaining two distributions change with T one should be able to answer the questions, "Does the shell ever set below the horizon?" and, "If so, does it rise again from below the horizon?" Of course, the horizon does not have any definite position because the Schwarzschild mass is an operator.

The internal geometrodynamics of the shell cannot by itself pose a more interesting question, namely, "If the shell rises again from below the horizon, *then where*?" To phrase that question, I would need to give some meaning to the *operator of spacetime geometry*. Of course, the spacetime geometry inside the shell should be Minkowskian and the spacetime geometry outside the shell should be described by the Schwarzschild line element in which M is replaced by the operator $\hat{M}$. However, in quantum theory one cannot say where the Minkowskian geometry ends and the Schwarzschild geometry begins because the position of the shell in the respective spacetimes is uncertain [we have seen that the location of the shell in the external Schwarzschild spacetime is also described by operators, namely, by $\hat{R}(\mathsf{T})$ and $\hat{T}(\mathsf{T})$]. It is thus conceptually quite tricky to define the operator of spacetime geometry.

I consider the minisuperspace shell model quite fascinating because it can help us to formulate and answer these and other relevant questions about the formation of a quantum black hole by the gravitational collapse of quantum matter.

ACKNOWLEDGMENTS

I thank the ITP at the University of Bern, the Center for Relativity and Gravitation at Penn State, and the ITP at UCSB for their hospitality during my sabbatical. Lively discussions with Petr Hájíček are gratefully acknowledged. This research was supported in part by the NSF grants PHY95-07719, PHY95-14240, and PHY94-07194, by Eberly Research Funds of Penn State, and by Swiss Nationalfonds and the Tomalla Foundation.

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